# A Theory of Second-Order Wireless Network Optimization and Its Application on AoI

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# Challenges with Current Methods

- Network Utility Maximization (NUM) showed success in finding optimal solutions for traditional objective functions.
- Current applications for real-time sensory estimation and video streaming use other metrics such as Age of Information (AoI).
- NUM techniques almost always fail when optimizing for newer performance metrics such as Age of Information (AoI) and Quality of Experience (QoE).
- NUM techniques such as primal-dual decomposition or Lyapunov drift-plus penalty **only capture** first order-statistics (i.e. mean).

# Our Approach: Second-Order Optimization

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- We can capture short-term system behavior by observing both the mean and temporal variance of random delivery processes.
    - Newer metrics involve higher-order behaviors to capture short-term network behavior.
- Given channel models, we derive the second-order capacity region consisting of all mean and temporal variance of packet delivery processes.
- We propose a new scheduling policy called Variance-Weighted Deficit (VWD), and prove it achieves every inner-point in the second-order capacity region.
- We apply VWD on an open problem: Optimizing AoI over independent Gilbert-Elliott channels.

# Wireless System Framework

- One Access Point (AP) serving $N$ clients referenced as $i = 1,2,\ldots,N$ at timeslots $t = 1,2,\ldots$.
- Each client is associated with an ON-OFF channel. The AP observes the channels at each timestep.
- AP can choose one client with an ON-channel for transmission at time $t$.

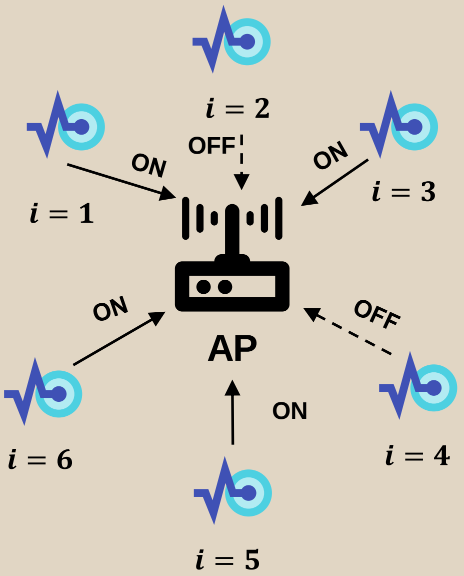


Fig. 1: AP serving six clients.

# Second-Order Model of Wireless Channels

- Given a subset of clients $S \subseteq \{1,2,\ldots,N\}$. Let $X_s(t)$ be the indicator function that at least one client has an ON channel at time $t$.
- Denote the channel mean rate of $S$ as

$$m_s := lim_{T\to\infty} \frac{\sum_{t=1}^{T} X_S(t)}{T}.$$

- Denote the channel temporal variance of $S$ as

$$v_S^2 := E[(\lim_{T\to\infty} \frac{\sum_{t=1}^{T} X_s(t) - T\, m_S}{\sqrt{T}})^2].$$

- Define the second-order channel model as a collection of all the mean and temporal variance of subsets $\{(m_S, v_S^2) \mid S \subseteq \{1,2,\ldots,N\}\}$.
  - How to define a client's mean rate and temporal variance?

# Second-Order Model for Application Performance

- Let $Z_i(t)$ be the indicator function that client $i$ receives a packet at time $t$.
- Define client $i$ service mean rate as $\mu_i \coloneqq lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_i(t)}{T}$.
- Define client $i$ service temporal variance as

$$\sigma_i^2 \coloneqq E[(\lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_i(t) - T\,\mu_i}{\sqrt{T}})^2].$$

- Utility of client $i$ is a function of $(\mu_i, \sigma_i^2)$, denoted by $\mathrm{F_i}(\mu_i, \sigma_i^2)$.
- Goal to maximize $\sum_{i=1}^{N} F_i(\mu_i, \sigma_i^2)$, given network constraints.
  - How to define the network constraints?

# Second-Order Model of Network Constraints

- Given a second-order channel model $\{(m_S, v_S^2) \mid S \subseteq \{1,2,\ldots,N\}\}$ at time $t$.

- Define the second-order capacity region as the set of all $\{(\mu_i, \sigma_i^2) \mid 1 \leq i \leq N\}$ such that a policy achieves
  - $lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_i(t)}{T} = \mu_i$.
  - $E[(\lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_i(t) - T\,\mu_i}{\sqrt{T}})^2] \leq \sigma_i^2$ for all clients $i = 1,2,\ldots,N$.

# Outer-Bound of Second-Order Capacity Region

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- **Theorem.** The second-order delivery model **can be in** the capacity region if the following holds
  - Sum of clients' mean rate in the subset $S$ **is less than or equal to** channel mean $\sum_{i \in S} \mu_i \leq m_S$ for all $S \subseteq \{1,2,\ldots,N\}$.
  - Sum of all clients' mean rate sum to the channel mean rate $\sum_{i=1}^{N} \mu_i = m_{\{1,2,\ldots,N\}}$.
  - Sum of clients' temporal variance square root is bigger or equal to channel temporal variance $\sum_{i=1}^{N} \sqrt{\sigma_i^2} \geq \sqrt{v_{\{1,2,\ldots,N\}}^2}$.
  - Client mean rate $\mu_i \geq 0$ for all $i$.

# Inner-Bound of Second-Order Capacity Region

- **Theorem.** The second-order delivery model **is in** the capacity region if the following holds
- Sum of clients' mean rate in the subset $S$ **is less than** channel mean rate $\sum_{i \in S} \mu_i < m_s$ for all $S \subsetneq \{1,2,\ldots,N\}$.
- Sum of all clients' mean rate sum to the channel mean rate $\sum_{i=1}^{N} \mu_i = m_{\{1,2,\ldots,N\}}$.
- Sum of clients' temporal variance is bigger or equal to the channel temporal variance $\sum_{i=1}^{N} \sqrt{\sigma_i^2} \geq \sqrt{v_{\{1,2,\ldots,N\}}^2}$.
- Client mean rate $\mu_i \geq 0$ and client temporal variance $\sigma_i^2 > 0$ for all $i$.
  - How to achieve this inner-bound?

# Variance-Weighted Deficit (VWD) Policy

- **Theorem.** VWD policy achieves every point in the inner bound of second-order capacity region.
- Given a point $(\mu_i, \sigma_i^2)$ in the bound.
- At time $t$, define client $i$ deficit as $d_i(t) = t\mu_i - \sum_{\tau=1}^{t} Z_i(\tau)$.
- From clients with ON channels, the controller picks the client with the largest $\frac{d_i(t-1)}{\sqrt{\sigma_i^2}}$.
- Since we are interested in AoI performance, how can we measure it for a policy such as VWD?

# Problem we Consider

- Optimize AoI by deriving the second-order model over Gilbert-Elliot channels.

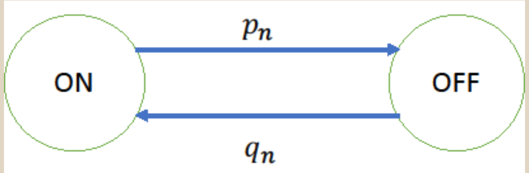



**Fig. 2: Gilbert-Elliot channel.**

- **Gilbert-Elliot channel:** two-state Markov process with transition probabilities $p_i$ and $q_i$.

- Each client generates an update with probability $\lambda_i$ and only keeps the newest update in memory.

- **Age of Information (AoI):** time difference between the newest information update at the source and the delivered information at the destination.

# Second-Order Model for Gilbert-Elliott Channels

- Under the Gilbert-Elliot channels, for all $S$

$$m_s = 1 - \prod_{i \in S} \frac{p_i}{p_i + q_i},$$

$$v_S^2 = 2 \sum_{k=1}^{\infty} \left( \prod_{i \in S} G_i(k+1) \prod_{i \in S} \frac{p_i}{p_i + q_i} \right) \prod_{i \in S} \frac{p_i}{p_i + q_i} + \prod_{i \in S} \frac{p_i}{p_i + q_i} - \left( \prod_{i \in S} \frac{p_i}{p_i + q_i} \right)^2,$$

with $G_i(k) = \frac{p_i}{p_i + q_i} + \frac{q_i}{p_i + q_i} (1 - p_i - q_i)^{k-1}$.

- Optimize AoI over Gilbert-Elliot channel model. How to express AoI using the second-order model?

# Second-Order Expression of AoI of $(\mu_i, \sigma_i^2)$

- Let $B_i(n)$ be the time between $n^{th}$ and $n+1^{th}$ deliveries.

- Long-term average AoI (theoretical AoI) $\overline{AoI_i}$ is given as $\overline{AoI_i} = \frac{E[B_i^2]}{2E[B_i]} + \frac{1}{\lambda_i} - \frac{1}{2}$.

- Estimate $\overline{AoI_i}$ from a Brownian motion random process $BM_{\mu_i,\sigma_i^2}(t)$.

- Approximate $B_i(n)$ by the amount of time the Brownian process increases by 1.

# Second-Order Expression of AoI of $(\mu_i, \sigma_i^2)$

- Therefore, we can approximate $\overline{AoI_i}$ (empirical AoI) as

$$\overline{AoI_i} \approx \frac{1}{2}\left(\frac{\sigma_i^2}{\mu_i^2} + \frac{1}{\mu_i}\right) + \frac{1}{\lambda_i} - \frac{1}{2}.$$

- Second-order optimization problem involves finding policy that maximizes $\sum_{i=1}^{N} F_i(\mu_i, \sigma_i^2)$.
  - For Gilbert-Elliott channels, what is the AoI performance function?

# Finding Clients' $(\mu_i, \sigma_i^2)$ for VWD

- With the goal of minimizing AoI over Gilbert-Elliot channels, we define objective function for client $i$ as $F_i(\mu_i, \sigma_i^2) = -\frac{1}{2}\left(\frac{\sigma_i^2}{\mu_i^2} + \frac{1}{\mu_i}\right) - \frac{1}{\lambda_i} + \frac{1}{2}$.
- We obtain the optimal delivery model for our VWD policy using steps:
  1. Find all sets of the second-order channel model $\{(m_s, v_s^2) \mid S \subsetneq \{1,2,\ldots,N\}\}$.
  2. Calculate client mean rates that satisfy $\sum_{i \in S} \mu_i \leq m_s - \delta$ and $\sum_{i=1}^{N} \mu_i = m_{\{1,2,\ldots,N\}}$.
  3. Client's temporal variance is lower bounded by channel temporal variance $\sum_{i=1}^{N} \sqrt{\sigma_i^2} \geq \sqrt{v_{\{1,2,\ldots,N\}}^2}$.
  4. $\mu_i \geq 0$ and $\sigma_i^2 > 0$ for all $i$.

# AoI Estimation of a Single Client

- Evaluate the theoretical AoI and empirical AoI on a single client.
- Results averaged over a 1000 independent runs.
- Each run contains 50000 timeslots.
- Empirical AoI is almost identical to the theoretical AoI.

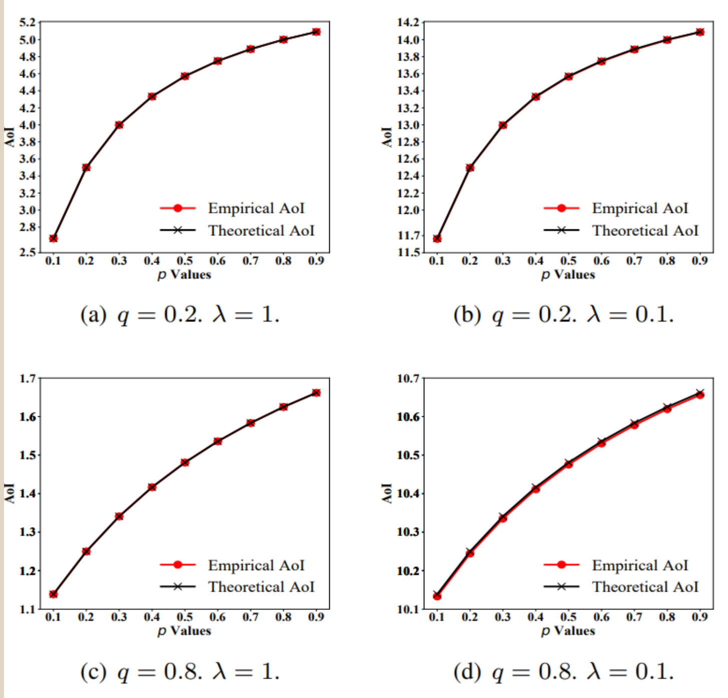
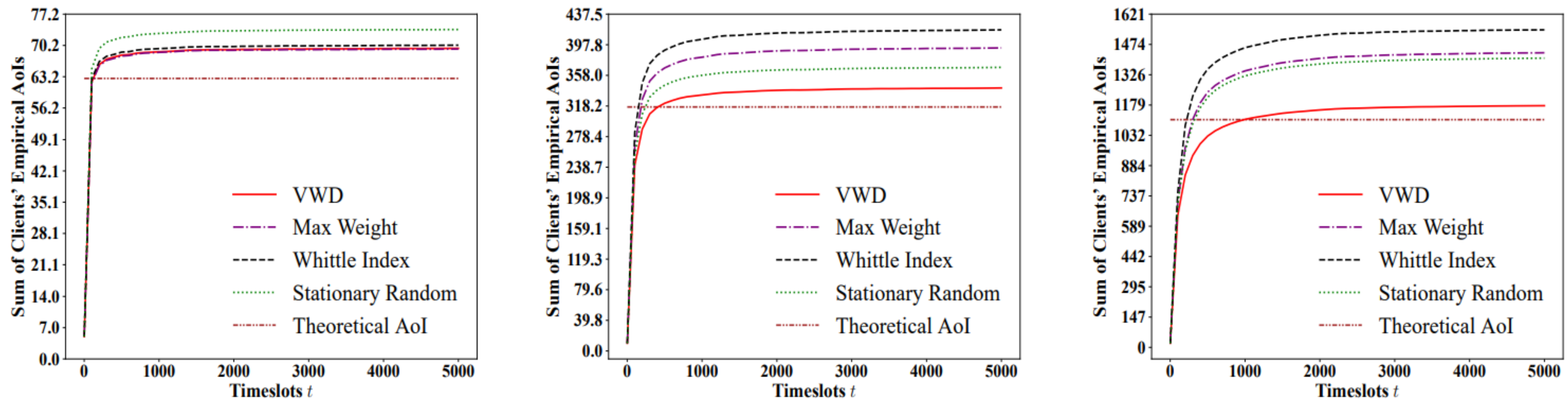



**Fig. 3: Model validation for a single client.**

# Simulations Setting

- Compare VWD against policies:
  - **Whittle index:** schedules the highest-indexed ON client $W_i(t) = \frac{AoI_i^2(t)}{2} - \frac{AoI_i(t)}{2} + \frac{AoI_i(t)}{q_i/(p_i+q_i)}$.
  - **Stationary randomized:** picks an ON client randomly proportional to $\mu_i$.
  - **Max weight:** picks an ON client with the largest $\frac{AoI_i(t)-z_i(t)}{\mu_i}$ with $z_i(t) = \frac{1}{\lambda_i}$.
- Simulations are ran for 1000 independent runs for 5000 timeslots.
- $\lambda_i$ randomly chosen from the range $(\frac{0.1}{N}, \frac{1}{N})$.
- Objective is to minimize $\sum_i \alpha_i \overline{AoI_i}$ , with client weight $\alpha_i$.

# AoI with Equal Weights' Results




(a) N = 5 Clients. (b) N = 10 Clients. (c) N = 20 Clients.

**Fig. 4: Uniform empirical AoI results averaged over 1000 runs.**

- Three different systems with: 5, 10, and 20 clients.
- VWD outperforms other policies for $N = 10$ and $N = 20$. VWD performs similar to Max weight for $N = 5$.
- VWD's empirical AoI is close to the theoretical AoI compared to other policies. VWD AoI approximation is accurate.

# Weighted AoI Results

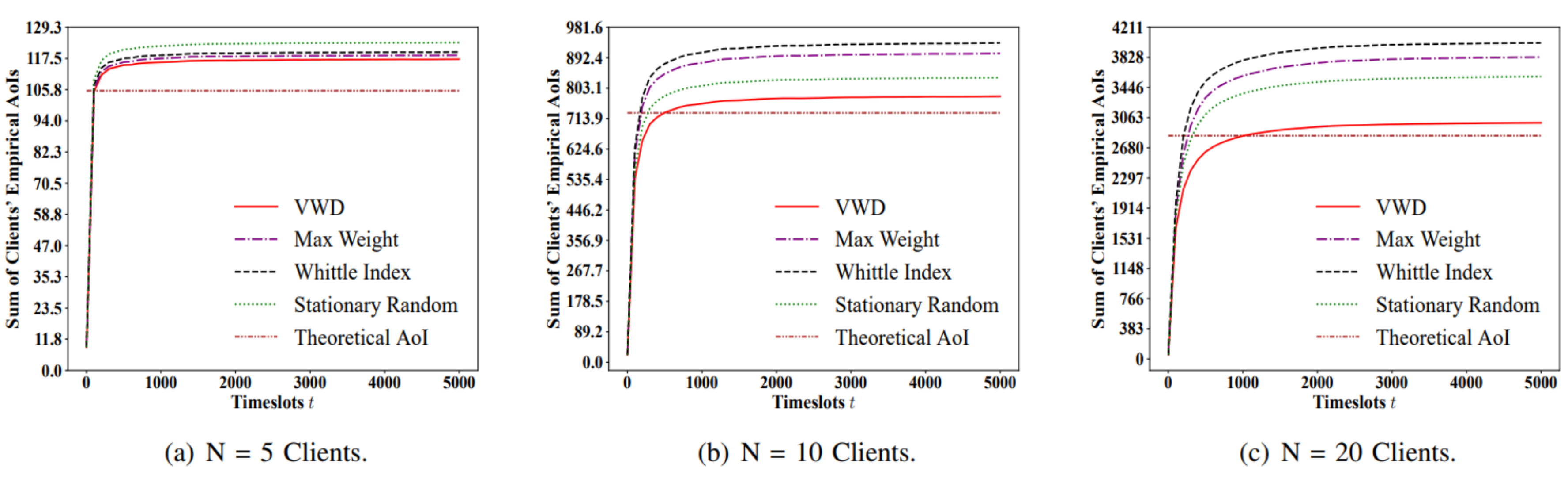



**Fig. 5: Weighted empirical AoI results averaged over 1000 runs.**

- Weights $\alpha_i$ were randomly selected from the range $(1,5)$.
- VWD outperforms other scheduling policies in the weighted AoI setting.
- VWD's empirical AoI is close to the theoretical AoI compared to other policies. VWD AoI approximation is accurate.

# Summary

- Proposed a new general model: the second-order capacity region for wireless networks.
- Introduced a new scheduling policy, VWD, that captures second-order statistics (temporal variance) within a second-order capacity region.
- Applied VWD on the unsolved optimization problem over Gilbert-Elliott channels.
- VWD outperforms other compared scheduling policies in both the weighted and unweighted AoI settings.

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